SOLUTION BY ALBERT A. BENNETT, University of Texas.

The system is homogeneous, so that if one solution exists, an infinite number exist. For convenience write c_p for $a_p + b_p$ and write in place of the inequalities the following equalities:

$$k_pc_p = \sum_{r=1}^{n} k_ra_r + e_p, \qquad e_p > 0, \qquad (p = 1, 2, 3, \cdots, n).$$

Solving for k_p, one has, provided that the denominators do not vanish,

$$k_p = \frac{e_p}{c_p} + \frac{1}{c_p}\frac{\sum_{r=1}^{n}\frac{e_ra_r}{c_r}}{1 - \sum_{r=1}^{n}\frac{a_r}{c_r}},$$

as may be verified at once by substitution.

Three cases occur.

1. $1 - \sum_{r=1}^{n} \frac{a_r}{c_r} > 0$. In this case, solutions are obtained readily by inserting arbitrary e's restricted only so that the k's shall be integers.

2. $1 - \sum_{r=1}^{n} \frac{a_r}{c_r} = 0$. There are no solutions in this case.

3. $1 - \sum_{r=1}^{n} \frac{a_r}{c_r} < 0$. In this case, also, there are no solutions. For if there were solutions, these would be in the form given above, and the expression, $\sum_{p=1}^{n} k_pa_p$, obtained by multiplying each solution, k_p by a_p and adding together would be

$$\sum_{p=1}^{n} k_pa_p = \sum_{p=1}^{n}\frac{e_pa_p}{c_p} + \sum_{p=1}^{n}\frac{a_p}{c_p}\frac{\sum_{r=1}^{n}\frac{e_ra_r}{c_r}}{1 - \sum_{r=1}^{n}\frac{a_r}{c_r}},$$
$$= \sum_{r=1}^{n}\frac{e_ra_r}{c_r}\frac{1}{1 - \sum_{r=1}^{n}\frac{a_r}{c_r}}.$$

Since the right-hand member would be negative under the hypothesis of this case, there could be no set of positive members, k_p, $p = 1, 2, \cdots, n$, of the form required for a solution. (The writer is indebted to Prof. H. P. MANNING for the treatment here given of this third case.)

2833 [1920, 227]. Proposed by W. H. ECHOLS, University of Virginia.

In *Engineering*, London, September 28, 1917, appeared the following equations concerning the stability of ships; they are employed by the naval constructors in the Norfolk (Virginia) Navy Yard, and they are of importance:

$$\tan^3\theta_0 + \frac{2m}{\rho}\tan\theta_0 - \frac{2x_0}{\rho} = 0, \qquad \tan^3(\theta_0 + \theta_1) + \frac{2m}{\rho}\tan(\theta_0 + \theta_1) - \frac{2x_0}{\rho} - \frac{2x}{\rho} = 0,$$

$$\tan^3(\theta_0 - \theta_2) + \frac{2m}{\rho}\tan(\theta_0 - \theta_2) - \frac{2x_0}{\rho} + \frac{2x}{\rho} = 0.$$

The required unknowns are θ_0, x_0 and m. The constants have values as follows: θ_1 and θ_2 are positive angles ranging from 15′ to 10°, x is positive and less than 5, and ρ is positive with a considerable range of values. A rapid solution involving small labor is desired, determining x_0 within the same limits as given for x.

SOLUTION BY THE PROPOSER.

It may be of interest to note that in the practical application of the problem, ρ is the metacentric radius when the vessel is upright, x is the shift of the center of gravity of the ship when a weight w is placed on the side at a distance d from the center line and is equal to wd/W, where W is the weight of displaced water. In a typical ship of 4000 tons, w equal to 2.5 tons, $\rho = 16$ ft.,

$x = 0.025$ ft., the constant $2x/\rho$ is 0.003125. The required m is the metacentric height, or the distance from the center of gravity of the ship to the metacenter when the vessel is upright. The "angle of heel," θ, is the angle through which the ship turns when the weight w is shifted transversely, and is small in the case of testing an ordinarily well designed ship. Under these circumstances the following solution appears to be valid.

Let

$$X = 2m/\rho, \qquad Y = 2x_0/\rho, \qquad a = 2x/\rho.$$

Then eliminating X and Y there results

$$\begin{vmatrix} \tan^3\theta_0, & \tan\theta_0, & 1 \\ \tan^3(\theta_0+\theta_1) - a, & \tan(\theta_0+\theta_1), & 1 \\ \tan^3(\theta_0-\theta_2) + a, & \tan(\theta_0-\theta_2), & 1 \end{vmatrix} = 0. \tag{1}$$

On expanding the determinant, (1) becomes

$$[\tan(\theta_0+\theta_1) - \tan\theta_0][\tan(\theta_0-\theta_2) - \tan\theta_0][\tan(\theta_0+\theta_1) - \tan(\theta_0-\theta_2)][\tan(\theta_0+\theta_1) + \tan(\theta_0-\theta_2) + \tan\theta_0] - a[\tan(\theta_0+\theta_1) + \tan(\theta_0-\theta_2) - 2\tan\theta_0] = 0. \tag{2}$$

In (2) put $z = \tan\theta_0$, $p = \tan\theta_1$, $q = \tan\theta_2$. On simplifying, the equation becomes

$$pq(p+q)\frac{(1+z^2)^2}{(1-pz)^2(1+qz)^2}\{pqz^3 + 2(p-q)z^2 - (3+2pq)z - (p-q)\} - a\{2pqz + (p-q)\} = 0, \tag{3}$$

a factor

$$\frac{1+z^2}{(1-pz)(1+qz)} \equiv K \tag{4}$$

being omitted since it does not involve the solution. Interchanging p and q in (3) merely changes the sign of z, therefore we may assume $p \geqq q$.

The angle of heel, θ_0, of the unweighted ship, due to faulty construction by which the material is not so distributed as to keep the center of gravity in a vertical plane of symmetry, is in general small. The expression represented by K in (4) is nearly unity under the conditions.

The roots of equation (3) are the abscissas of the points of intersection of the straight line

$$y = 2apqz + a(p-q), \tag{5}$$

and the curve

$$y = pq(p+q)K^2\{pqz^3 + 2(p-q)z^2 - (3+2pq)z - (p-q)\}. \tag{6}$$

The curve (6) cuts the z-axis in the same points as does the cubic

$$y = pq(p+q)\{pqz^3 + 2(p-q)z^2 - (3+2pq)z - (p-q)\}, \tag{7}$$

and it has vertical asymptotes $z = 1/p$ and $z = -1/q$. The ordinates of (7) at $z = -1/q$ and $z = 1/p$ are respectively

$$+\,pq(p+q)\left(\frac{p}{q^2} + \frac{1}{q} + p + q\right) \qquad \text{and} \qquad -\,pq(p+q)\left(\frac{q}{p^2} + \frac{1}{p} + p + q\right).$$

The cubic has three real roots γ, α, β in the respective intervals

$$-\infty, \qquad -1/q, \qquad +1/p, \qquad +\infty,$$

and cuts the y-axis at $y = -pq(p^2 - q^2)$. It has a concavo-convex inflection at

$$z = -\frac{2}{3}\frac{p-q}{pq},$$

the ordinate there being positive. The middle root α lies between the origin and the z-intercept of the tangent at $z = 0$, which is

$$z = -\frac{p-q}{3+2pq}.$$

The straight line (5) cuts the z-axis midway between the asymptotes at a point

$$A = -(p-q)/2pq.$$

The slope of (5) is positive and this line (as we shall show) cuts (6) in only one point between the asymptotes. The abscissa of this point is the root we seek.

Represent by Q the cubic cofactor of K^2 in (6). Take the logarithm of both sides of that equation and differentiate. Then

$$\frac{1}{2y}\frac{dy}{dz} = \frac{2z}{1+z^2} + \frac{(p-q)+2pqz}{(1-pz)(1+qz)} + \frac{1}{2Q}\frac{dQ}{dz}. \tag{8}$$

The ordinate y is positive between A and α, so also is Q. The first term on the right is negative when z is negative. The derivative DQ is negative from $z = 0$ to its negative root which is easily seen to be less than $-1/q$. The inflection of $y = Q$ occurs at $(4/3)A$.

The numerator obtained by adding the first two terms on the right is

$$(p-q) + 2(1+pq)z - (p-q)z^2.$$

This is clearly negative for $z = -(p-q)/[2(1+pq)]$ and all smaller values, and therefore from A to $-\frac{1}{2}(p-q)$, inclusive. Hence the derivative Dy is negative throughout this interval.

Consider the interval from $-\frac{1}{2}(p-q)$ to α. The derivative DQ decreases in absolute value as x varies from $-\frac{1}{2}(p-q)$ to 0. The greatest value of Q in the interval from $-\frac{1}{2}(p-q)$ to α is at $-\frac{1}{2}(p-q)$. The numerical value of the last term in (8) is greater than that of DQ taken at $z = 0$ divided by twice that of Q at $-\frac{1}{2}(p-q)$, or

$$\frac{3+2pq}{(p-q)\{1+2pq+(1-\frac{1}{4}pq)(p-q)^2\}},$$

which is greater than

$$\frac{3}{p(1+2p^2)}.$$

The second term on the right in (8) is 0 at A and increases (the numerator increasing and the denominator decreasing) through positive values from $z = A$ to $z = 0$ when it is greatest and equal to $p - q$. Therefore this term is less than p throughout the interval from $-\frac{1}{2}(p-q)$ to α. The sum of the last two terms on the right in (8) is certainly negative for values of p which make $p^2(1+2p^2) < 3$, which is true if $p < 1$, or $\theta_1 < 45°$.

These results show that y in (6) is a one-valued continuously decreasing function of z throughout the interval A to α and therefore there is only one root of (3) in this interval.

The conditions of the problem are such that the required root of (3) is small; a first and practically sufficient approximation

$$z_1 = -\frac{p-q}{pq}\,\frac{a+pq(p+q)}{2a+(p+q)(3+2pq)}, \tag{9}$$

is obtained by making K provisionally equal to 1 and neglecting terms in Q containing powers of z above the first. A new approximation can be obtained by putting z_1 for z in K and in z^3 and z^2 in Q, and solving again for z.

It is of interest to note that the original equations can be written

$$Y = \alpha X + \alpha^3, \qquad Y + a = \beta X + \beta^3, \qquad Y - a = \gamma X + \gamma^3,$$

regarding α, β, γ as variable parameters. These represent three straight lines which envelope, respectively, the semi-cubic parabolas

$$27Y^2 + 4X^3 = 0, \qquad 27(Y+a)^2 + 4X^3 = 0, \qquad 27(Y-a)^2 + 4X^3 = 0.$$

These straight lines make given angles with each other. The second makes $+\theta_1$ with the first and the third makes $-\theta_2$ with the first.

A graphical solution of the equations consists in fitting a figure composed of three straight lines, meeting in a point and making given angles with each other, so that each is tangent to the corresponding semi-cubic parabola. The intersection of these three tangents has the required coördinates X, Y. The slope of the tangent to the first parabola is the required $\tan\theta_0$.

An actual navy yard experiment gave the data

$$\theta_1 = 4°\,49'\,14'', \qquad \rho = 2.63, \qquad \theta_2 = 4°\,39'\,36'', \qquad x = 0.0568.$$

Application of the computations indicated above gives $p = 0.0843$, $q = 0.0815$, $a = 0.0432$, $\tan\theta_0 = -0.03124$, $\theta_0 = -(1°\,47'\,21'')$. $X = 0.511$, $Y = -0.016$, $m = 1.344$, $x_0 = -0.021$.